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A weak solution for a class of the elliptic systems with even functional

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ABSTRACT We get one result which shows the existence of at least one weak solution for a class of the elliptic systems involving subcritical Sobolev exponents nonlinear term with even functional on the bounded domain with smooth boundary. We get this result by variational method and critical point theory.

Key Words and Phrases: A class of elliptic systems, subcritical Sobolev exponents nonlinear term, even functional, variational method, critical point theory.

AMS 2000 Mathematics Subject Classifications: 35J50, 35J55

1. Introduction

Let Ω be a bounded domain of R^n with smooth boundary, $n \geq 3$, α, β, γ, p, q are real constants. In this paper we investigate existence of weak solutions for the following class of the elliptic systems with Dirichlet boundary condition

$$\begin{cases} -\Delta u &= \alpha u + \beta v + F_u(u,v) \quad \text{in } \Omega, \\ -\Delta v &= \beta u + \gamma v + F_v(u,v) \quad \text{in } \Omega, \\ u &= v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here

$$F(u,v) = \frac{2}{p+q}|u|^p|v|^q,$$

where p, $q > 1$ are real constants, $2 < p+q < 2^*$, $2^* = \frac{2n}{n-2}$.

In this paper we consider a class of elliptic systems involving subcritical Sobolev exponents nonlinear term with even functional. Since the pioneering work on the subject in [1], these problem have been investigated in many ways. For a survey on the scalar case we recommend the paper [2] and the references therein. For the system case we recommend the paper [3].

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Indeed weak solutions of (1.1) correspond to critical points of the continuous and *Frechét* differentiable functional

$$\begin{aligned} I(u,v) &= \frac{1}{2}\int_\Omega [|\nabla u|^2+|\nabla v|^2-\alpha u^2-2\beta uv-\gamma v^2]dx-\int_\Omega \frac{2}{p+q}|u|^p|v|^q dx \\ &= Q_{\alpha,\beta,\gamma}(u,v)-\int_\Omega \frac{2}{p+q}|u|^p|v|^q dx, \end{aligned}$$

where $Q_{\alpha,\beta,\gamma}(u,v)=\frac{1}{2}\int_\Omega [|\nabla u|^2+|\nabla v|^2-\alpha u^2-2\beta uv-\gamma v^2]dx$. The crucial point in this paper is that the nonlinear term of (1.1) has subcritical exponents $2<p+q<\frac{2n}{n-2}$. When $2<p+q<\frac{2n}{n-2}$, the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^{p+q}(\Omega)$ is compact, where $W_0^{1,2}(\Omega)$ is a Sobolev space, so we can assure that the associated functional of (1.1) satisfies the $(P.S.)$ condition.

Let $\lambda_1<\lambda_2\le\cdots\le\lambda_k\le\cdots$ be eigenvalues of the eigenvalue problem $-\Delta u=\lambda u$ in Ω, $u=0$ on $\partial\Omega$, and ϕ_k be eigenfunctions belonging to eigenvalues λ_k, $k\ge 1$. Let $W_0^{1,2}(\Omega)$ be a Sobolev space with the norm

$$\|u\|^2_{W_0^{1,2}(\Omega)}=\int_\Omega |\nabla u|^2 dx.$$

Let $E=W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)$ be a Hilbert space endowed with the norm

$$\|(u,v)\|_E^2=\|u\|^2_{W_0^{1,2}(\Omega)}+\|v\|^2_{W_0^{1,2}(\Omega)}.$$

Let A be $\begin{pmatrix}\alpha & \beta\\ \beta & \gamma\end{pmatrix}\in M_{2\times 2}(R)$. Let us set

$$\begin{aligned} H_{\lambda_i} &= \text{span}\{\phi_i|\ -\Delta\phi_i=\lambda_i\phi_i\}, \\ q_{\lambda_i}(\alpha,\beta,\gamma) &= \text{Det}(\lambda_i I-A)=(\lambda_i-\alpha)(\lambda_i-\gamma)-\beta^2. \end{aligned}$$

Let $\eta^1_{\lambda_i}$ and $\eta^2_{\lambda_i}$ be eigenvalues of the matrix $\begin{pmatrix}\lambda_i-\alpha & -\beta\\ -\beta & \lambda_i-\gamma\end{pmatrix}\in M_{2\times 2}(R)$, i. e.,

$$\eta^1_{\lambda_i}=\frac{1}{2}\{2\lambda_i-\gamma-\alpha-\sqrt{((2\lambda_i-\gamma-\alpha))^2-4q_{\lambda_i}(\alpha,\beta,\gamma)}\},$$

$$\eta^2_{\lambda_i}=\frac{1}{2}\{2\lambda_i-\gamma-\alpha+\sqrt{(2\lambda_i-\gamma-\alpha)^2-4q_{\lambda_i}(\alpha,\beta,\gamma)}\}.$$

We are looking for weak solutions of (1.1) in E. The weak solutions in E satisfies

$$\begin{aligned} &\int_\Omega [(-\Delta u)z+(-\Delta v)w-\alpha uz-\beta vw-\beta uz-\gamma vw]dx \\ &-\int_\Omega [\frac{2p}{p+q}|u|^{p-1}|v|^q z+\frac{2q}{p+q}|u|^p|v|^{q-1}w]dx=0 \end{aligned}$$

for all $(z,w)\in E$.

Our main result is as follows:

Theorem 1.1. *Assume that p, $q>1$ are real constants, $2<p+q<2^*$, $2^*=\frac{2n}{n-2}$, and α, β are real constants satisfying the following:,*

(i) $\alpha > 0$, $\beta > 0$, $\gamma < 0$, $-\gamma > \alpha$,
(ii) $q_{\lambda_i}(\alpha,\beta,\gamma) = det\begin{pmatrix} \lambda_i - \alpha & -\beta \\ -\beta & \lambda_i - \gamma \end{pmatrix} < 0 \quad$ *for* $1 \le i \le 2m$, $m \ge 1$
and
(iii) $q_{\lambda_i}(\alpha,\beta,\gamma) > 0$, $\forall i \ge 2m+1$.
Then (1.1) has at least one nontrivial weak solution.

For the proof of Theorem 1.1 we use variational method and critical point theory. In Section 2, we introduce eigensubspaces spanned by eigenfunctions and prove that the corresponding functional of (1.1) satisfies the $(P.S.)$ condition. In Section 3, we obtain some variational linking inequalities of $I(u,v)$ and prove Theorem 1.1.

2. Variational approach on eigenspace

Let $(c^1_{\lambda_i}, d^1_{\lambda_i})$ and $(c^2_{\lambda_i}, d^2_{\lambda_i})$ be the eigenvectors of $\begin{pmatrix} \lambda_i - \alpha & -\beta \\ -\beta & \lambda_i - \gamma \end{pmatrix} \in M_{2\times 2}(R)$ corresponding to $\eta^1_{\lambda_i}$ and $\eta^2_{\lambda_i}$ respectively. Let us set

$$\begin{aligned}
D_{\lambda_i} &= \{(\alpha,\beta,\gamma) \in R^3 |\ q_{\lambda_i}(\alpha,\beta,\gamma) < 0 \text{ for } 1 \le i \le 2m,\ m \ge 1, \\
&\quad\ q_{\lambda_i}(\alpha,\beta,\gamma) > 0,\ \forall i \ge 2m+1\}, \\
D'_{\lambda_i} &= D_{\lambda_i} \cap \{-\gamma \le \alpha\}, \\
D''_{\lambda_i} &= D_{\lambda_i} \cap \{-\gamma > \alpha\}, \\
E_{\lambda_i} &= \{(c\phi, d\phi) \in E |\ (c,d) \in R^2, \phi \in H_{\lambda_i}\}, \\
E^1_{\lambda_i} &= \{(c^1_{\lambda_i}\phi, d^1_{\lambda_i}\phi) \in E |\ \phi \in H_{\lambda_i}\}, \\
E^2_{\lambda_i} &= \{(c^2_{\lambda_i}\phi, d^2_{\lambda_i}\phi) \in E |\ \phi \in H_{\lambda_i}\}, \\
H^+(\alpha,\beta,\gamma) &= (\oplus_{\eta^1_{\lambda_i}>0} E^1_{\lambda_i}) \oplus (\oplus_{\eta^2_{\lambda_i}>0} E^2_{\lambda_i}), \\
H^-(\alpha,\beta,\gamma) &= (\oplus_{\eta^1_{\lambda_i}<0} E^1_{\lambda_i}) \oplus (\oplus_{\eta^2_{\lambda_i}<0} E^2_{\lambda_i}), \\
H^0(\alpha,\beta,\gamma) &= (\oplus_{\eta^1_{\lambda_i}=0} E^1_{\lambda_i}) \oplus (\oplus_{\eta^2_{\lambda_i}=0} E^2_{\lambda_i}).
\end{aligned}$$

Then $H^+(\alpha,\beta,\gamma)$, $H^-(\alpha,\beta,\gamma)$ and $H^0(\alpha,\beta,\gamma)$ are the positive, negative and null space relative to the quadratic form $Q_{\alpha,\beta,\gamma}(u,v)$ in E. Because $(\lambda_i - \alpha)(\lambda_i - \gamma) - \beta^2 \neq 0$,

$$H^0(\alpha,\beta,\gamma) = \{0\}.$$

Lemma 2.1. *Assume that p, $q > 1$ are real constants, $2 < p+q < 2^*$, $2^* = \frac{2n}{n-2}$ and the conditions (i), (ii) and (iii) of Theorem 1,1 hold. Let $(\alpha,\beta,\gamma) \in R^3$. Then*
(i) $E^1_{\lambda_i}$ and $E^2_{\lambda_i}$ are eigenspace for the operator $M_{\alpha\beta\gamma}$, $M_{\alpha\beta\gamma}(u,v) = (-\Delta u - \alpha u - \beta v, -\Delta v - \beta u - \gamma v)$ associated with $Q_{\alpha\beta\gamma}$ with eigenvalues $\frac{\eta^1_{\lambda_i}}{\lambda_i}$ and $\frac{\eta^2_{\lambda_i}}{\lambda_i}$ respectively.
(ii) $E^1_{\lambda_i}$ and $E^2_{\lambda_i}$ generate E.

(iii) Let $i \geq 1$. Then we have that

$$\lim_{(\alpha,\beta,\gamma)\to(\alpha_0,\beta_0,\gamma_0)} \eta^1_{\lambda_i}(\alpha,\beta,\gamma) = \eta^1_{\lambda_i}(\alpha_0,\beta_0,\gamma_0)$$

and

$$\lim_{(\alpha,\beta,\gamma)\to(\alpha_0,\beta_0,\gamma_0)} \eta^2_{\lambda_i}(\alpha,\beta,\gamma) = \eta^2_{\lambda_i}(\alpha_0,\beta_0,\gamma_0).$$

uniformly with respect to $i \in N$.

Proof. The proof can be obtained by easy computations. ∎

Let us define

$$C_{p,q}(\Omega) = \inf_{(u,v)\in E\setminus(0,0)} \frac{\int_\Omega(|\nabla u|^2 + |\nabla v|^2)dx}{(\int_\Omega |u|^p|v|^q dx)^{\frac{2}{p+q}}} \quad \text{for } (u,v) \in E. \tag{2.1}$$

LEMMA 2.2. *Assume that p, $q > 1$ are real constants, $2 < p+q < 2^*$ and the conditions (i), (ii), (iii) of Theorem 1.1 hold. Then if $\|(u_n,v_n)\|_E \to \infty$ and $(u_n,v_n)_n$ is a sequence such that*

$$\frac{\int_\Omega[(\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q, \frac{2q}{p+q}|u_n|^p|v_n|^{q-1})\cdot(u_n,v_n) - \frac{4}{p+q}|u_n|^p|v_n|^q]dx}{\|(u_n,v_n)\|_E} \longrightarrow 0,$$

then there exist $(u_{h_n}, v_{h_n})_n$ and $(z,w) \in E$ such that

$$\frac{(\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q, \frac{2q}{p+q}|u_n|^p|v_n|^{q-1})}{\|(u_n,v_n)\|_E} \to (z,w) \in E \qquad \frac{(u_{h_n}, v_{h_n})}{\|(u_{h_n}, v_{h_n})\|_E} \to (0,0).$$

Proof. We note that

$$\begin{aligned}
&\int_\Omega[\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1})v_n]dx - \frac{4}{p+q}\int_\Omega |u_n|^p|v_n|^q dx \\
&\leq \int_\Omega[\frac{2p}{p+q}|u_n|^p|v_n|^q + \frac{2q}{p+q}|u_n|^p|v_n|^q)]dx - \frac{4}{p+q}\int_\Omega |u_n|^p|v_n|^q dx \\
&\leq (\frac{2p}{p+q} + \frac{2q}{p+q} - \frac{4}{p+q})\int_\Omega |u_n|^p|v_n|^q dx \\
&\leq C_{p,q}^{-\frac{2}{p+q}}(\Omega)(\frac{2p}{p+q} + \frac{2q}{p+q} - \frac{4}{p+q})\|(u_n,v_n)\|_E^{p+q}, \qquad 2 < p+q < 2^*.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|\frac{\int_\Omega[\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n]dx}{\|(u_n,v_n)\|_E}\|_{L^r} \\
\leq\ & C_{p,q}^{-\frac{2}{p+q}}(\Omega)(\frac{2p}{p+q} + \frac{2q}{p+q})\|\|(u_n,v_n)\|_E^{p+q-1}\|_{L^r} \\
\leq\ & C(\frac{\|(u_n,v_n)\|_E^{p+q}}{\|(u_n,v_n)\|_E})^{\frac{p+q-1}{p+q}}\|(u_n,v_n)\|^l,
\end{aligned}$$

where $l=-1+\frac{p+q-1}{p+q}<0$. When $2<p+q<\frac{2n}{n-2}$, the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^{p+q}(\Omega)$ is compact. Thus there exist $(u_{h_n},v_{h_n})_n$ such that

$$\frac{\int_\Omega[\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q u_{h_n}+\frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1}v_{h_n}]dx}{\|(u_{h_n},v_{h_n})\|_E} \tag{2.2}$$
$$=\int_\Omega(\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q,\frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1})\cdot\frac{(u_{h_n},v_{h_n})}{\|(u_{h_n},v_{h_n})\|_E}dx\longrightarrow 0.$$

It follows that there exists $(z,w)\in E$ such that

$$\frac{(\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q,\frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1})}{\|(u_{h_n},v_{h_n})\|_E}\to(z,w)\in E,\qquad \frac{(u_{h_n},v_{h_n})}{\|(u_{h_n},v_{h_n})\|_E}\to(0,0).$$

■

Lemma 2.3. *((P.S.)condition)*
Assume that p, $q>1$ are real constants, $2<p+q<2^*$ and the conditions (i), (ii), (iii) of Theorem 1.1 hold. Let $i\in N$ and $(\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}$. Then there exist a neighborhood W of $(\alpha_0,\beta_0,\gamma_0)$ such that for any $(\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}}D'_{\lambda_i}$, the functional $I(u,v)$ satisfies $(P.S.)$ condition on E.

Proof. Let $i\in N$, $(\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}$ and W be a neighborhood of $(\alpha_0,\beta_0,\gamma_0)$. Let $(\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}}D'_{\lambda_i}$. Let $c\in R$ and $(u_n,v_n)_n\subset E$ be a sequence such that $I(u_n,v_n)\to c$ and $DI(u_n,v_n)\to\theta$, $\theta=(0,0)$. We claim that $(u_n,v_n)_n$ is bounded. By contradiction we suppose that $\|(u_n,v_n)\|_E\to\infty$ and set $(\hat{u_n},\hat{v_n})=\frac{(u_n,v_n)}{\|(u_n,v_n)\|_E}$. Since $(\hat{u_n},\hat{v_n})_n$ is bounded, up to a subsequence, $(\hat{u_n},\hat{v_n})_n$ converges weakly to some $(\hat{u},\hat{v})$ in E. Let $(\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}}D'_{\lambda_i}$. Since $DI(u_n,v_n)\to 0$, we have

$$(-\Delta-A)(\hat{u_n},\hat{v_n})-\langle\frac{\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n+\frac{2q}{p+q}|u_n|^p|v_n|^{q-1})v_n}{\|(u_n,v_n)\|_E},(\hat{u_n},\hat{v_n})\rangle\longrightarrow 0. \tag{2.3}$$

Since $DI(u_n,v_n)\to 0$ and $I(u_n,v_n)\to c$, we also have

$$\frac{DI(u_n,v_n)\cdot(u_n,v_n)}{\|(u_n,v_n)\|}$$
$$=\frac{2I(u_n,v_n)}{\|(u_n,v_n)\|_E}-\frac{\int_\Omega(\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n+\frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n-\frac{4}{p+q}|u_n|^p|v_n|^q)dx}{\|(u_n,v_n)\|_E}\longrightarrow 0.$$

Thus we have

$$\frac{\int_\Omega(\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n+\frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n-\frac{4}{p+q}|u_n|^p|v_n|^q)dx}{\|(u_n,v_n)\|_E}\longrightarrow 0. \tag{2.4}$$

By Lemma 2.2, (2.2) and (2.4), there exist a sequence $(u_{h_n},v_{h_n})_n$ such that

$$\frac{\int_\Omega[\frac{2p}{p+q}|u_{h_n}|^{p-1}|v_{h_n}|^q u_{h_n}+\frac{2q}{p+q}|u_{h_n}|^p|v_{h_n}|^{q-1})v_{h_n}]dx}{\|(u_{h_n},v_{h_n})\|_E}\longrightarrow 0$$

and

$$\frac{(u_{h_n}, v_{h_n})}{\|(u_{h_n}, v_{h_n})\|_E} \to (0,0).$$

Thus we have $(\hat{u}, \hat{v}) = (0,0)$, which is absurd because $\|(\hat{u}, \hat{v})\|_E = 1$. Thus $(u_n, v_n)_n$ is bounded. Thus $(u_n, v_n)_n$ converges weakly to some(u, v). Let $P_- : E \to H^-(\alpha, \beta, \gamma) = \oplus_{\mu^1_{\lambda_i}<0,\ 1\le i\le 2m} E^1_\lambda$ and $P_+ : E \to H^+(\alpha, \beta, \gamma) = (\oplus_{\mu^2_{\lambda_i}>0, 1\ \le i\le 2m} E^2_{\lambda_i}) \oplus (\oplus_{\mu^1_{\lambda_i}>0,\ i\ge 2m+1} E^1_{\lambda_i})$ denote the orthogonal projections. We claim that (u_n, v_n) converges to $(u, v) \in E = H^-(\alpha, \beta, \gamma) \oplus H^+(\alpha, \beta, \gamma)$ strongly. Since $DI(u_n, v_n) \to (0,0)$, we have

$$\langle DI(u_n, v_n), (u_n, v_n)\rangle = \int_\Omega [(-\Delta u_n)u_n + (-\Delta v_n)v_n - \alpha u_n^2 - \beta v_n v_n - \beta u_n v_n - \gamma v_n^2]dx$$
$$- \int_\Omega (\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n)dx \longrightarrow 0.$$

Since (u_n, v_n) converges to (u, v) weakly, we have

$$\begin{aligned}
&\lim_{n\to\infty} \langle DI(u_n, v_n), (u_n, v_n)\rangle \\
&= \lim_{n\to\infty} (\|P_+(u_n, v_n)\|_E^2 - \|P_-(u_n, v_n)\|_E^2) \\
&- \lim_{n\to\infty} (\int_\Omega (\frac{2p}{p+q}|u_n|^{p-1}|v_n|^q u_n + \frac{2q}{p+q}|u_n|^p|v_n|^{q-1}v_n)dx) \\
&= \lim_{n\to\infty} (\|P_+(u_n, v_n)\|_E^2 - \|P_-(u_n, v_n)\|_E^2) \\
&- \int_\Omega (\frac{2p}{p+q}|u|^{p-1}|v|^q u + \frac{2q}{p+q}|u|^p|v|^{q-1}v)dx = 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{n\to\infty} (\|P_+(u_n, v_n)\|_E^2 - \|P_-(u_n, v_n)\|_E^2) \\
&= \int_\Omega (\frac{2p}{p+q}|u|^{p-1}|v|^q u + \frac{2q}{p+q}|u|^p|v|^{q-1}v)dx.
\end{aligned}$$

Thus $\lim_{n\to\infty}(\|P_+(u_n, v_n)\|_E^2 - \|P_-(u_n, v_n)\|_E^2)$ converges strongly to $\|P_+(u, v)\|_E^2 - \|P_-(u, v)\|_E^2$. Thus $\lim_{n\to\infty} \|(u_n, v_n)\|^2 = \lim_{n\to\infty}(\|P_+(u_n, v_n)\|_E^2 + \|P_-(u_n, v_n)\|_E^2) = \|P_+(u, v_n)\|_E^2 - \|P_-(u, v)\|_E^2$. Thus $(u_n, v_n)_n$ converges strongly to (u, v) such that

$$DI(u, v) = \lim_{n\to\infty} DI(u_n, v_n),$$

so (u, v) is a critical point of I. ∎

3. Proof of Theorem 1.1

We shall find critical points of the functional $I(u, v) \in C^{1,1}(E, R)$,

$$\begin{aligned}
I(u, v) &= \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx \qquad (3.1) \\
&= Q_{\alpha,\beta,\gamma}(u, v) - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx,
\end{aligned}$$

where $Q_{\alpha,\beta,\gamma}(u,v) = \frac{1}{2}\int_\Omega[|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx$. We note that

$$\begin{aligned} H^-(\alpha,\beta,\gamma) &= ((\oplus_{1\le j\le 2m} E^1_{\lambda j}), \\ H^+(\alpha,\beta,\gamma) &= ((\oplus_{1\le j\le 2m} E^2_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^1_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^2_{\lambda j})). \end{aligned}$$

Let us set

$$\begin{aligned} S_\sigma &= \{(u,v)\in E|\ \|(u,v)\|_E = \sigma\}, \\ B_\sigma &= \{(u,v)\in E|\ \|(u,v)\|_E \le \sigma\}, \\ Q &= \bar{B_R}\cap H^-(\alpha,\beta,\gamma) \oplus \{r(u_0,v_0)|\ 0<r<R, \\ &\quad (u_0,v_0)\in (\oplus_{1\le j\le 2m} E^2_{\lambda j}) \oplus (\oplus_{j\ge 2m+1} E^1_{\lambda j}),\ Q_{\alpha,\beta,\gamma}(u_0,v_0) = 1\}. \end{aligned}$$

LEMMA 3.1. *Assume that p, $q>1$ are real constants, $2<p+q<2^*$ and the conditions (i), (ii), (iii) of Theorem 1.1 hold. Let $i\in N$ and $(\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}$. Then there exist a neighborhood W of $(\alpha_0,\beta_0,\gamma_0)$ such that for any $(\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}$, there exists a constant $\sigma>0$ such that*

$$\inf_{(u,v)\in S_\sigma\cap H^+(\alpha,\beta,\gamma)} I(u,v) > 0, \qquad \inf_{B_\sigma\cap H^+(\alpha,\beta,\gamma)} I(u,v) > -\infty,$$

where B_σ is a ball centered at (0,0) with radius $\sigma>0$.

Proof. Let (α,β,γ) be any element of $W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}$. Let $(u,v)\in H^+(\alpha,\beta,\gamma)$. Then we have

$$Q_{\alpha,\beta,\gamma}(u,v) = \frac{1}{2}\int_\Omega[|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx \ge \frac{1}{2}\min\{\frac{\eta^2_{\lambda_1}}{\lambda_1}, \frac{\eta^1_{\lambda_{2m+1}}}{\lambda_{2n+1}}\}\|(u,v)\|^2_E.$$

By (2.1), we have

$$\begin{aligned} I(u,v) &= Q_{\alpha,\beta,\gamma} - \frac{2}{p+q}\int_\Omega |u|^p|v|^q dx \\ &\ge \frac{1}{2}\min\{\frac{\eta^2_{\lambda_1}}{\lambda_1}, \frac{\eta^1_{\lambda_{2m+1}}}{\lambda_{2n+1}}\}\|(u,v)\|^2_E - \frac{2}{p+q}(C_{p,q}(\Omega))^{-\frac{p+q}{2}}\|(u,v)\|^{p+q}_E. \end{aligned}$$

Since $2<p+q<\frac{2n}{n-2}$ and $\min\{\frac{\eta^2_{\lambda_1}}{\lambda_1}, \frac{\eta^1_{\lambda_{2m+1}}}{\lambda_{2n+1}}\}>0$, there exists a small constant $\sigma>0$ such that if $(u,v)\in S_\sigma\cap H^+(\alpha,\beta,\gamma)$, then $I(u,v)>0$. Moreover if $(u,v)\in B_\sigma\cap H^+(\alpha,\beta,\gamma)$, then $I(u,v) > -\frac{2}{p+q}(C_{p,q}(\Omega))^{-\frac{p+q}{2}}\|(u,v)\|^{p+q}_E > -\infty$. ■

LEMMA 3.2. *Assume that p, $q>1$ are real constants, $2<p+q<2^*$ and the conditions (i), (ii), (iii) of Theorem 1.1 hold. Let $i\in N$ and $(\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}$. Then there exist a neighborhood W of $(\alpha_0,\beta_0,\gamma_0)$ such that for any $(\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}}$ D'_{λ_i},*
there exist a constant $R>0$ and an element $(u_0,v_0)\in(\oplus_{1\le j\le 2m} E^2_{\lambda j})\oplus(\oplus_{j\ge 2m+1} E^1_{\lambda j})$

with $Q_{\alpha,\beta,\gamma}(u_0,v_0)=1$ *such that if* $(u,v)\in\partial Q=\partial(\bar{B_R}\cap H^-(\alpha,\beta,\gamma)\oplus\{r(u_0,v_0)|\ 0<r<R,\ (u_0,v_0)\in(\oplus_{1\le j\le 2m}E^2_{\lambda j})\oplus(\oplus_{j\ge 2m}E^1_{\lambda j}),\ Q_{\alpha,\beta,\gamma}(u_0,v_0)=1\}$, *then*

$$\sup_{(u,v)\in\partial Q} I(u,v)<0,\qquad \sup_{(u,v)\in Q} I(u,v)<\infty.$$

Proof. Let (α,β,γ) be any element of $W\backslash\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}}D'_{\lambda_i}$. Let us choose an element $(u_0,v_0)\in(\oplus_{1\le j\le 2m}E^2_{\lambda j})\oplus(\oplus_{j\ge 2m+1}E^1_{\lambda j})$ and $(u,v)\in H^-(\alpha,\beta,\gamma)\oplus\{r(u_0,v_0)|\ r>0,\ (u_0,v_0)\in(\oplus_{1\le j\le 2m}E^2_{\lambda j})\oplus(\oplus_{j\ge 2m+1}E^1_{\lambda j}),\ Q_{\alpha,\beta,\gamma}(u_0,v_0)=1\}$. Then (u,v) is of the form $(u,v)=(x,y)+r(u_0,v_0)$, $(x,y)\in H^-(\alpha,\beta,\gamma)$ and $(u_0,v_0)\in\partial B_1\cap(\oplus_{1\le j\le 2m}E^2_{\lambda j})\oplus(\oplus_{j\ge 2m+1}E^1_{\lambda j})$ with $Q_{\alpha,\beta,\gamma}(u_0,v_0)=1$, $r>0$. Then we have

$$\begin{aligned}I(t(u,v)) &= \frac{t^2}{2}\int_\Omega[|\nabla u|^2+|\nabla v|^2-\alpha u^2-2\beta uv-\gamma v^2]dx-\frac{2t^{p+q}}{p+q}\int_\Omega|u|^p|v|^q dx\\ &= \frac{t^2}{2}\int_\Omega[|\nabla x|^2+|\nabla y|^2-\alpha x^2-2\beta xy-\gamma y^2]dx+\frac{t^2}{2}r^2\\ &\quad -\frac{2t^{p+q}}{p+q}\int_\Omega|x+ru_0|^p|y+rv_0|^q dx\\ &\le \frac{t^2}{2}r^2-\frac{2t^{p+q}}{p+q}\int_\Omega|x+ru_0|^p|y+rv_0|^q dx.\end{aligned}$$

Because $Q_{\alpha,\beta,\gamma}(x,y)<0$ if $(x,y)\in H^-(\alpha,\beta,\gamma)$. Since $2<p+q$, $I(t(u,v))\to-\infty$ as $t\to\infty$. Thus there exists a large number $R>0$ such that if $(u,v)\in\partial Q=\partial(\bar{B_R}\cap H^-(\alpha,\beta,\gamma)\oplus\{r(u_0,v_0)|\ 0<r<R,\ (u_0,v_0)\in(\oplus_{1\le j\le 2m}E^2_{\lambda j})\oplus(\oplus_{j\ge 2m+1}E^1_{\lambda j}),\ Q_{\alpha,\beta,\gamma}(u_0,v_0)=1\})$, then $I(u,v)<0$. Thus we have $\sup_{(u,v)\in\partial Q}I(u,v)<0$. Moreover if $(u,v)\in Q$, then $I(u,v)<\frac{R^2}{2}<\infty$. Thus $\sup_{(u,v)\in Q}I(u,v)<\infty$.

Proof of Theorem 1.1

Let $i\in N$, $(\alpha,\beta,\gamma)\in W\backslash\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}}D'_{\lambda_i}$. We note that the functional I is continuous and *Fréchet* differentiable on E. By Lemma 2.3, $I(u,v)$ satisfies the $(P.S.)$ condition. By Lemma 3.1 and Lemma 3.2, there exist constants $\sigma>0$ and $R>0$ such that

$$\inf_{(u,v)\in S_\sigma\cap H^+(\alpha,\beta,\gamma)}I(u,v)>0,\qquad \inf_{B_\sigma\cap H^+(\alpha,\beta,\gamma)}I(u,v)>-\infty$$

and

$$\sup_{(u,v)\in\partial Q} I(u,v)<0,\qquad \sup_{(u,v)\in Q} I(u,v)<\infty.$$

Let us define

$$\Gamma=\{\gamma\in C(\bar{Q},E)|\ \gamma=id \text{ on } \ \partial Q\}$$

and

$$c=\inf_{h\in\Gamma}\sup_{(u,v)\in Q}I(h(u,v)).$$

By the classical Deformation Lemma (cf. Theorem A.4 in [4]), c is a critical point of $I(u,v)$ such that

$$0 < \inf_{(u,v)\in S_\sigma \cap H^+(\alpha,\beta,\gamma)} I(u,v) < c = I(u,v) < \sup_{(u,v)\in Q} I(u,v).$$

Thus (1.1) has at least one nontrivial solution.

Competing Interests

The authors declare that they have no competing interests.

Authors's contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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